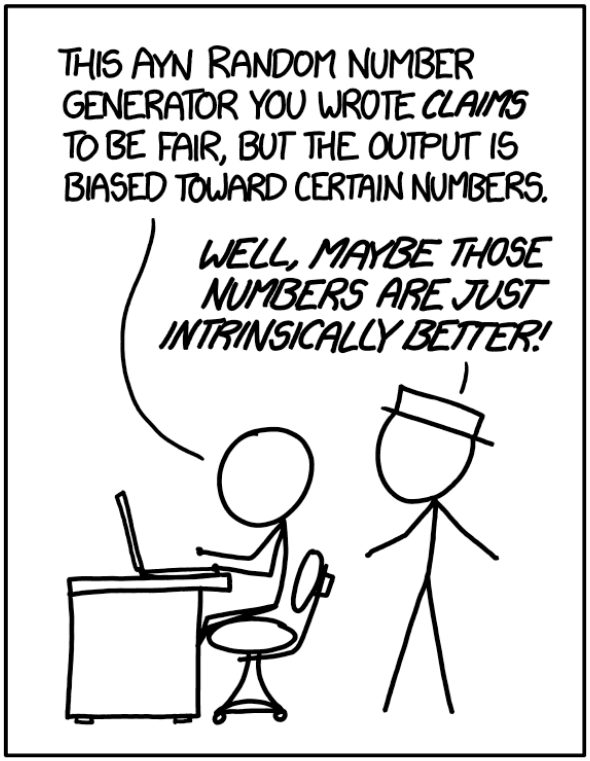
Stat 88: Probability & Math. Stat in Data Science



https://xkcd.com/1277

Lecture 7: 1/31/2024

Finish up base rate fallacy, Random variables & their distributions

Finish chapter 2, and 3.1, 3.2

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Warm up: Please answer at pollev.com/shobhana

For the questions below, we have that $0 < P(A), P(B) < 1$

1) True or false: If A and B are mutually exclusive, they must be independent.

2) True or false: If $B \subset A$, then A and B must be independent.

3) True or false: If A and B are independent, then so are their complements.

4) If A, B, C are mutually independent events with $P(A) = 0.3$, $P(B) = 0.8$, $P(C) = 0.4$, what is the probability that at least one of these events occurs (that is, find $P(A \cup B \cup C)$).

Agenda

- Base rate fallacy
- The Monty Hall Problem
- Random variables and their distributions
- The binomial distribution

Recall Harvard study of physicians

- Harvard study: 60 physicians, students, and house officers at the Harvard Medical school were asked the following question:
- "If a test to detect a disease whose ***prevalence*** is 1/1,000, has a false positive rate of 5 per cent, what is the chance that a person found to have a positive result actually has the disease, assuming that you know nothing about the person's symptoms or signs?"
- *Prevalence* aka *Base Rate* = **fraction of population that has disease**.
- *False positive rate*: fraction of positive results among people who don't have the disease
- *Positive result*: test is positive
- Note that the test is assumed to be accurate in the sense that it will never give a false negative, so prob of positive test given disease is 100%.
- We saw that P(disease | pos test) ≈ 2% (≈ 1/51)

Base Rate Fallacy

- $P(D \mid \text{pos. test})$ or ***posterior*** probability =
- Recall that ***prior*** probability of disease = 0.001 = 0.1%
- **P(+ test) = P(+ & disease) + P(+ & no disease)** (since either you have the disease or not, so we have a partition of the event "positive test")
- Base rate fallacy: Ignore the base rate and focus only on the likelihood. (Moral of this story: ignore the base rate at your own peril)
- Note: Want $P(D \mid +)$ but most people focus on the test giving correct results for negative tests 95% of the time, that is P(no disease| neg)
- What happens to the posterior probability if we change the prior probability?

Monty Hall problem

There are 3 doors, A, B, C, behind one is a new car (a Ferrari, say), and behind the other two are goats. Now suppose you are the contestant, and you choose door A. Then the host, Monty Hall, opens one of the other two doors, say B, to show you a goat!

He asks you if you want to switch to C or stick with your original choice A. What should you do?

Counting review

- Recall the product rule of counting, where we counted number of outcomes when we had a sequence of k actions, each with n_i outcomes, so the total number of outcomes is $n_1 \times n_2 \times \ldots n_k$

- For examples, we want to count the number of sequences of 3 letters taken from the English alphabet without replacement.

- Suppose we don't care about the sequence but just *which* letters were chosen (so *abc* = *bca* = *cab* etc.) Then all of these combinations count as 1 selection. We need to take the number we got above and divide by the number of arrangements of 3 letters.

- $\binom{n}{k}$ = number of ways to choose a subset of size k out of a set of size n.

Section 3.1: Vocabulary

- When we have two kinds of tickets in a box and we draw tickets at random from this box, each draw is called a ***trial***
- We call the two kinds (binary) of outcomes ***Success***, and ***Failure***

- Might be with replacement (like a coin toss) or without replacement (drawing voters from a city and checking number of mask mandate supporters)

- Read about Paul the octopus and Mani the parakeet and their soccer predictions
- Note that Paul made 8 correct 2010 WC predictions. What is the chance of 8 correct if picking completely at random? (like tossing a coin and getting all heads)

Back to counting outcomes of tosses

- Toss a coin 8 times, how many possible outcomes?

- What is the chance of ***all*** heads?

- If each of the students in this class present today flip a coin 8 times, what is the chance that ***at least 1 person*** gets all heads?

3.2 Random Variables

- A real number – we don't know exactly *what* value it will take, but we know the possible values.

- The number of heads when a coin is tossed 3 times could be 0, 1, 2, or 3.
- The sum of spots when a pair of dice is rolled could be 2, 3, 4, 5, …, 12.

- These are both examples of *random variables*.
- *Variable* because the number takes different values
- *Random variable* because the outcomes are not certain.

Random variables

- Using random variables helps to write events more clearly and concisely.

- It is a way to ***map*** the function space Ω to real numbers
- For example: Let X represent the number of heads in 3 tosses.

- We can write down the ***distribution*** of X, which consists of its possible values and their probabilities.

- The function describing the distribution is called the ***probability mass function*** ***(*** $f(x)$***)***
- Note that the probabilities must add up to 1.
- We can visualize it using a *probability histogram*.

Random variables, distribution table & histogram

- For example: Let X represent the **number of heads in 3 tosses**.
- We can write down the ***distribution*** of X, which consists of the possible values of X and the probabilities of X taking these values & make a histogram:

Outcome	X(outcome)	probability

- The function describing the distribution is called the ***probability mass function*** $f(x)$, where $\boldsymbol{f(x) = P(X = x)}$

Another example

- Let X be the **sum of spots** when a pair of dice is rolled.
- Write down the probability distribution table of X :

	2	3	4	5	6	7	8	9	10	11	12

- Probability histogram:

Random Variables

- Note that even if two random variables have the same distribution, they are not necessarily equal. For example, let X be the number of heads in 2 tosses of a fair coin, and Y be the number of tails.
- That is, we can talk about the *particular* values being equal and *distributions* being equal – and these are not the same thing.

3.3 The Binomial distribution

- Many situations can be modeled using the following set up:
  - We have a ***fixed*** number of ***independent*** trials, each of which has ***two*** possible outcomes. "success"(S) and "failure"(F)
  - The probability of success stays ***constant*** from trial to trial.

- Example: toss a coin 10 times, count the number of heads
  - Each toss is an independent trial
  - A success is a head.
  - P(success) = 0.5

- Need to specify number of trials (***n***), and P(success) (***p***)
  - Example: number of people who accept credit card offer from bank
  - Number of aces in 10 rolls of a die.

Binomial distribution: Example

- Consider a box with **one** red ball and **eleven** blue ones.

- One draw is made. What is the probability that the ball is red?
  - $n = 1, p = 1/12$
  - $P(R) = 1/12$

- Now 4 draws are made, ***with replacement***. What is the probability that ***exactly*** 1 draw is red (out of the 4)?
  - Notice that this is like a tossing a coin 4 times, with P(head) = 1/12.

- $P(RBBB) =$
- How many such sequences are there?
- What is the probability of all such sequences (with1 R, 3B)?

Binomial distribution: Example

- What if we want to compute the probability of **2** red balls in 4 draws? We need the number of sequences of R and B that have 2 R and 2 B.
- $P(RRBB) =$
- There are 6 such sequences (how?), so if we let X = # of red balls in 4 draws with replacement, we have that

$$P(X = 2) = \binom{n}{k} \times p^2 \times (1 - p)^2$$

where $p = P(red)$

- We say that X has the **Binomial distribution with parameters n and p**, and write it as $X \sim Bin(n, p)$ if X takes values $0, 1, \ldots, n$ and

$$P(X = k) = \binom{n}{k} \times p^k \times (1 - p)^{n-k}$$

Characteristics of the binomial distribution

- There are n trials, where n is FIXED beforehand.
- The chance (p) of a success stays the SAME from trial to trial
- Each trial results in either success (S) or failure (F)
- The trials are INDEPENDENT of each other.
- $X \sim Bin(n, p)$, possible values of X: $0, 1, 2, \ldots, n$
- Use python to compute numerical values of probabilities (read section in text, in 3.3)

Identifying binomial random variables

Which of the following are binomial random variables?

- Number of heads in 12 tosses of a fair coin.
- Number of tosses until we see two heads.
- Number of queens in a five card hand
- Number of Democrats in a simple random sample of 500 adult voters drawn from the SF Bay Area.

Vocabulary

- When we have two kinds of tickets in a box and we draw tickets at random from this box, each draw is called a ***trial***
- We call the two kinds (binary) outcomes ***Success***, and ***Failure***

- Might be with replacement (like a coin toss) or without replacement (taking a simple random sample of residents and checking number of people who watched Ryan Cochran-Siegle win a silver yesterday.)

- Random variables (usually denoted by X, Y etc) are numbers that ***map*** the function space Ω to real numbers, so they inherit a probability distribution.

- Probability distribution of a random variable X, is a description of the values taken by X, and the probabilities that X takes these values.

- The ***probability mass function*** of X, denoted by $f(x)$, is a function that gives, for each value x taken by X, its chance $P(X = x)$.