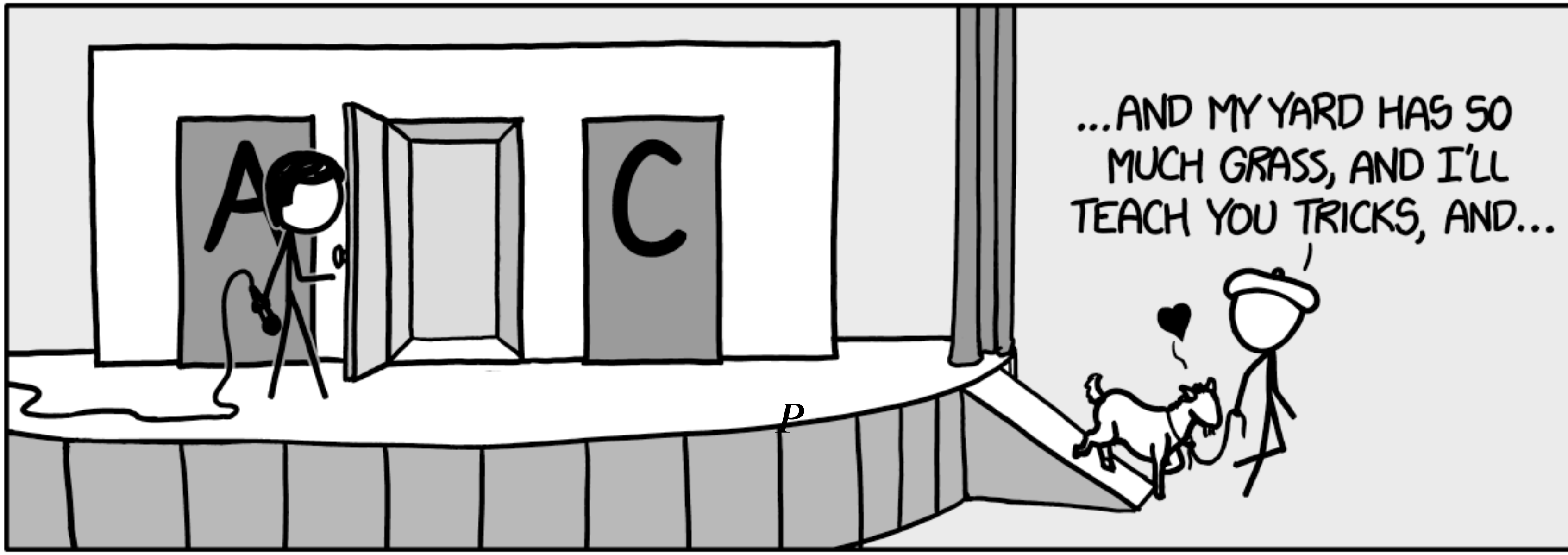
Stat 88: Probability and Statistics in Data Science



https://xkcd.com/1282/

Lecture 5: 01/26/2024

Multiplication rule, Symmetry in Sampling, Bayes' Rule

Sections 2.1, 2.2, 2.3

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Warm up

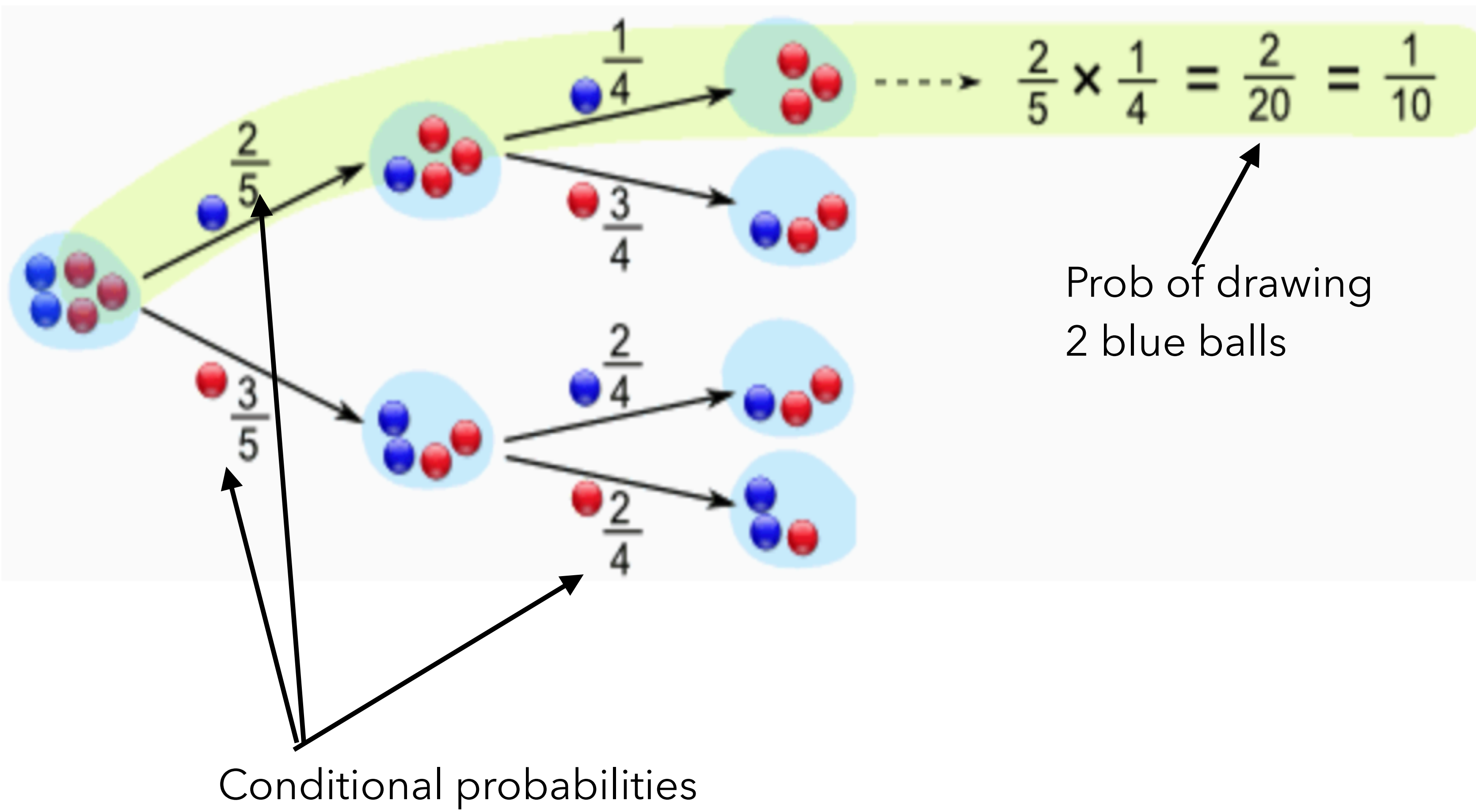
1. What is the probability that the top card in a standard 52 card deck is a queen ***and*** the bottom card is a queen?
2. What is the probability that the top card in a standard 52 card deck is a queen ***or*** the bottom card is a queen?
3. There are 3 doors, A, B, C, behind one is a new car (a Ferrari, say), and behind the other two are goats. Now suppose you are the contestant, and you choose door A. Then Monty Hall opens one of the other two doors, say B, to show you a goat!

He asks you if you want to switch to C or stick with your original choice A, you say…?

Agenda

- Multiplication rule, division rule
- Definition of independence
- Product rule in counting (remember Ozzy Osprey and his outfits?)
- Symmetries in sampling & counting
- Bayes’ rule, and maybe…
- Use and interpretation of Bayes’ rule

Drawing balls from an urn



Multiplication rule

- Let $A,\ B \subseteq \Omega,\ P(A) > 0, P(B) > 0$
- Conditional probability written as $P(B|A)$, read as "the probability of the event B, given that the event A has occurred"
- The probability that two things will both happen is the chance that the first happens, *multiplied* by the chance that the second will happen *given* that the first has happened.
- Multiplication rule for the ***intersection*** of two events A and B:

$$P(A \cap B) = P(A|B)P(B)$$

$$P(B \cap A) = P(B|A)P(A)$$

Therefore, $P(A|B)P(B) = P(B|A)P(A)$

- Division rule: $P(A|B) = \dfrac{P(AB)}{P(B)}, P(B) \neq 0$

Multiplication rule

$$P(A \cap B) = P(A \mid B) \times P(B)$$

- Ex.: Draw a card at random, from a standard deck of 52
  - P(King of hearts) =?

- Draw **2** cards one by one, **without** replacement.
  - P(1st card is K of hearts)=

  - P(2nd card is Q of hearts| 1st is K of hearts) =

  - P(1st card is K of hearts AND 2nd is Q of hearts) =

Independence

- Let $A,\ B \subseteq \Omega,\ P(A) > 0, P(B) > 0$
- What if $P(B \mid A) = P(B)$? That is, knowing that A occurred made no difference to the probability of B.
- In this case, we say that A and B are ***independent*** of each other, that is, they do not affect each other's probabilities.
- To check if two events A and B are independent, you can check if:

$$P(A \mid B) = P(A) \text{ or}$$

$$P(B \mid A) = P(B) \text{ or}$$

$$P(AB) = P(A) \times P(B)$$

- Die rolls are independent, cards dealt from top of deck are not.

Mutually exclusive vs Independent

- Make sure you understand the difference; these are very different ideas, though both apply to pairs of events
- If two events are mutually exclusive, this means that the occurrence of one prevents the occurrence of the other. (This means that it reduces the chance of the other occurring to 0.)
- If two events are independent, this means that the occurrence of one does not change the chance of the other occurring.
- Do NOT assume independence without justification.

Case of Sally Clarke and SIDS: Was this justice? Or quite the opposite?

- Around 2003, Sally Clark, in a famous murder trial had two children one year apart who both died mysteriously. Sally Clarke's defence was that the babies both died of Sudden Infant Death Syndrome (SIDS)
- $A =$ event the first child dies of SIDS
- $B =$ event the second child dies of SIDS.
- Assumption: $P(A) = P(B) = 1/8543$ (based on stats, unconditional probability)
- Read 2.5.4

Back to the addition rule:

- Addition rule: If *A* and *B* are ***mutually exclusive*** events, then the probability that **at least one** of the events will occur is the sum of their probabilities:

$$P(A \cup B) = P(A) + P(B)$$

- If they are not mutually exclusive, does this still hold?
- How do we write the event that "at least one of the events *A* or *B* will occur? How do we draw it?

Inclusion-Exclusion Formula (general addition rule)

- $P(A \cup B) = P(A) + P(B) - P(AB)$

- $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

- (Draw a Venn diagram)

Inclusion-Exclusion Formula (general addition rule)

- Of course, if A and B (or A and B and C) *don't* intersect, then the general addition rule becomes the **simple** addition rule of

$$P(A \cup B) = P(A) + P(B), \ or$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Back to the second warm up problem

- What is the probability that the top card in a standard 52 card deck is a queen ***or*** the bottom card is a queen?

A *partition* of Ω

- We say that $A_1, A_2, \ldots A_n$ form a partition of Ω if they are mutually exclusive of each other and their union is Ω.
- Example: A and A^c form a partition of Ω.

Example from text

- Deal 5 cards from the top of a well shuffled deck. What is the probability that all are hearts? (Extend the multiplication rule)

- Deal 5 cards, what is the chance that they are all the same suit? (flush)

Sec. 2.2: Symmetry in Simple Random Sampling

- One of the topics we will revisit many times is ***simple random sampling***.
- Sampling **without** replacement, each time with equally likely probabilities
- Example to keep in mind: dealing cards from a deck

- Sampling with replacement: We keep putting the sampled outcomes back before sampling again. (Can do this to simulate die rolls or coin tosses)

- When we want to count the number of possible outcomes from repeating an action such as sampling, we will use the product rule of counting.

Product rule of counting

- Recall Ozzy Osprey with 4 pairs of pants, and 3 t-shirts.
- If a set of actions (call them $A_1, A_2, \ldots, A_n$) can result, respectively, in $k_1, k_2, \ldots, k_n$ possible outcomes, then the entire set of actions can result in:

$$k_1 \times k_2 \times k_3 \times \ldots \times k_n \text{ possible outcomes}$$

- For example: toss a coin twice. Each toss can have 2 possible outcomes, therefore 2 tosses can have 4 possible outcomes.

How many ways to arrange…

- Consider the box that contains O R A N G E:
- How many ways can we rearrange these letters?

- Now say we only want to choose **2 letters** out of the six: __ __

Symmetries in cards

- Deal 2 cards from top of the deck.
    - How many possible sequences of 2 cards?
    - What is the chance that the second card is red?

- P(5th card is red)

- $P(R_{21} \cap R_{35})$ is the prob that 21st card and 35th cards are red.

- P(7th card is a queen)

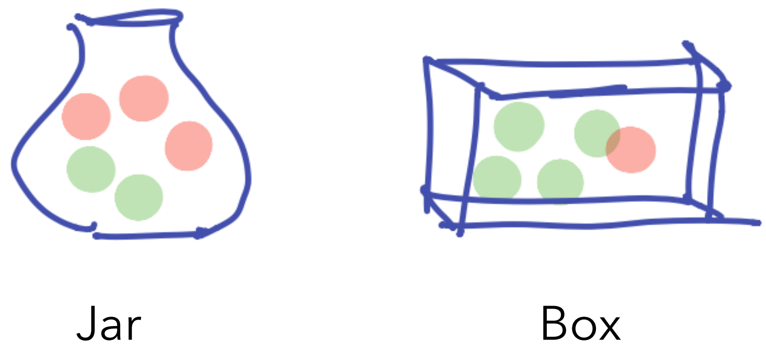
- $P(B_{52} \mid R_{21}R_{35})$

Exercise

- The English language has 26 letters. 5 letters are chosen **with** replacement. What is the chance that the *middle* three letters are all *different*, and the *first* and *last* are the *same* as each other, and also the *same* as one of the three middle letters.

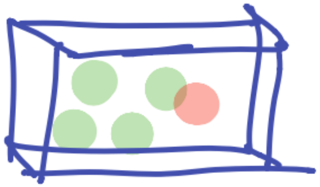
Section 2.3: Bayes' Rule:

- I have two containers: a jar and a box. Each container has five balls: The jar has three red balls and two green balls, and the box has one red and four green balls.
- Say I pick one of the containers at random, and then pick a ball at random. What is the chance that I picked the box, if I ended with a red ball?



- Let B be the event of picking the box, J the event of picking the jar
- Let R, G be the events of drawing a red ball and a green ball, respectively.

Jars and boxes



$P(B) = P(J) = 1/2; P(R \mid B) = ?, P(R \mid J) = ?$

Prior and Posterior probabilities

- The ***prior*** probability of drawing the box = ___ (before we knew anything about the balls drawn)

- The ***posterior*** probability of drawing the box = ____ (this is after we *updated* our probability, *given* the information about which ball was drawn)

Computing Posterior Probabilities: Bayes' Rule

- We want the *posterior* probability. That is, the conditional prob for the first stage A, ***given*** the second stage B.
- Division rule (for conditional probability) =
- Using the multiplication rule on $P(A \cap B)$, we get:
- Rule first written down by Rev. Thomas Bayes in the 18th century. Helps us compute posterior probability, given prior prob. And **likelihoods** (which are conditional probabilities for the *second* stage given the first, which are generally easier to compute.)

2.4: Use and interpretation of Bayes' rule

- Harvard study: 60 physicians, students, and house officers at the Harvard Medical school were asked the following question:
- "If a test to detect a disease whose ***prevalence*** is 1/1,000, has a false positive rate of 5 per cent, what is the chance that a person found to have a positive result actually has the disease, assuming that you know nothing about the person's symptoms or signs?"
- *Prevalence* aka *Base Rate* = fraction of population that has disease.
- *False positive rate*: fraction of positive results among people who don't have the disease
- *Positive result*: test is positive

- What is your guess – without any computations?

Tree diagram for disease and positive test

- $P(D|\text{positive test})$ or *posterior* probability =
- Recall that prior probability = 0.001 = 0.1%

Base Rate Fallacy

- $P(D \mid \text{pos. test})$ or *posterior* probability =
- Recall that prior probability = 0.001 = 0.1%

- P(+ test) = P(+ & disease) + P(+ & no disease) (since either you have the disease or not, so we have a partition of the event "positive test")

- Base rate fallacy: Ignore the base rate and focus only on the likelihood. (Moral of this story: ignore the base rate at your own peril)

- Note: Want $P(D \mid +)$ but most people focus on the test giving correct results for negative tests 95% of the time, that is P(no disease| neg)

- What happens to the posterior probability if we change the prior probability?

Let's make a deal!: The Monty Hall Problem

There are 3 doors, A, B, C, behind one is a new car (a Ferrari, say), and behind the other two are goats.

Now suppose you are the contestant, and you choose door A. Then Monty Hall opens one of the other two doors, say B, to show you a goat!

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