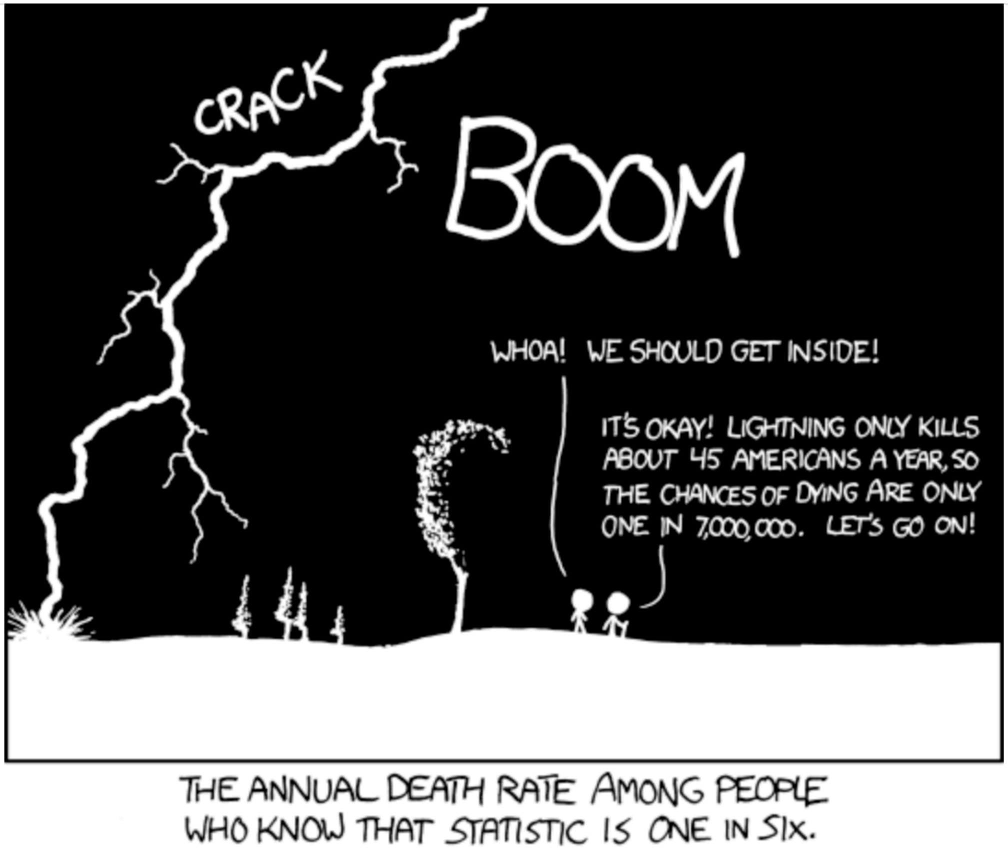
Stat 88: Probability and Statistics in Data Science



https://xkcd.com/795/

Lecture 3: 1/25/2022

Axioms of Probability, Intersections,

Sections 1.3, 2.1

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Agenda

Quick recap of terms

Section 1.2: Exact calculations or bounds

- unions vs intersections
- addition rule

Section 1.3: Fundamental Rules (the Axioms of Probability)

- Notation
- Axioms
- Consequences of the axioms
- De Morgan's Law

Terminology

- *Experiment*: action that results in exactly one of several possible outcomes or results, and chance or randomness is involved - that is, each time we perform the action, the outcome will be different, and we don't know exactly which outcome will occur.
- A collection of all possible outcomes of an action is called a ***sample space*** or an ***outcome space***. Usually denoted by Ω (sometimes also by *S*).
- An ***event*** A is a collection of outcomes and $A \subset \Omega$
- A *distribution* of the outcomes over some categories represents the proportion of outcomes in each category (each outcome appears in one and only one category)
- The *complement* of an event A is an event consisting of all the outcomes that are not in A. It is denoted by A^C and we have that $P(A^C) = 1 - P(A)$ (*Complement Rule)*

Terminology & rules

- $P(A) = \dfrac{\#(A)}{\#(\Omega)}$

- Multiplication: If an experiment is in k stages, and each stage i results in n_i outcomes, then the total number of outcomes is $n_1 \times n_2 \times \ldots n_k$

- The complement of an event A is an event consisting of all the outcomes that are not in A. It is denoted by A^C and we have that $P(A^C) = 1 - P(A)$ (Complement Rule)

- $P(A \mid B) = \dfrac{\#(A \text{ and } B)}{\#(B)}$ (The conditional probability of A given B)

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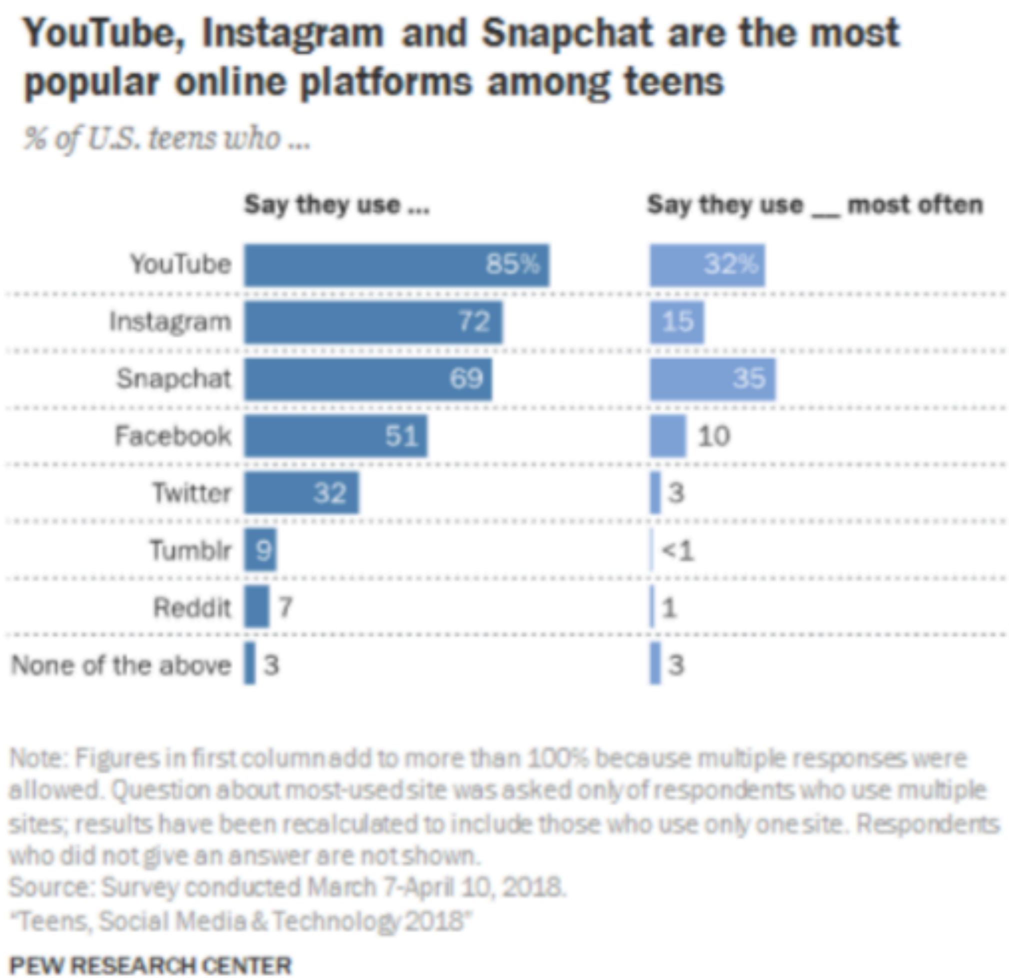
From Friday: Not equally likely outcomes



1. What is the chance that a randomly picked teen uses FB most often?

0.1

2. What is the chance that a randomly picked teen did *not* use FB most often?

0.9

3. What is the chance that FB *or* Twitter was their favorite?

$0.1+0.03 = 0.13$

4. What is the chance that the teen used FB, just not most often?

$0.51-0.1 = 0.41$

5. Given that the teen used FB, what is the chance that they used it most often?

$10/51 = 0.1/0.51 \approx 0.2$

Exercise from Friday

A six-sided fair die is rolled twice:

- If the first roll lands on 1, what is the chance that the second roll lands on a number bigger than 2?

Exercise: Find the probability that the second number is greater than the twice the first number.

Rules that we used: Addition rule

If all the possible outcomes are *equally likely*, then each outcome has probability ***1/n***, where ***n*** = number of possible outcomes.

If an event A contains k possible outcomes,

then ***P(A) = k/n***.

Probabilities are between 0 and 1

If two events A and B don't overlap, then the probability of A or B = P(A) + P(B) (since we can just add the number of outcomes in one and the other, and divide by the number of outcomes in Ω)

Rules of probability

Let's think about what rules we can lay down, based on what we have seen so far.

Origins of probability: de Méré's paradox

Questions that arose from gambling with dice.

Antoine Gombaud,
Chevalier de Méré

Blaise Pascal

Pierre de Fermat

The dice players
Georges de La Tour
(17th century)

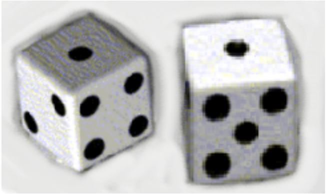
De Méré's Paradox: in section tomorrow

We can think about probability as a numerical measure of uncertainty, and we will define some basic principles for computing these numbers.

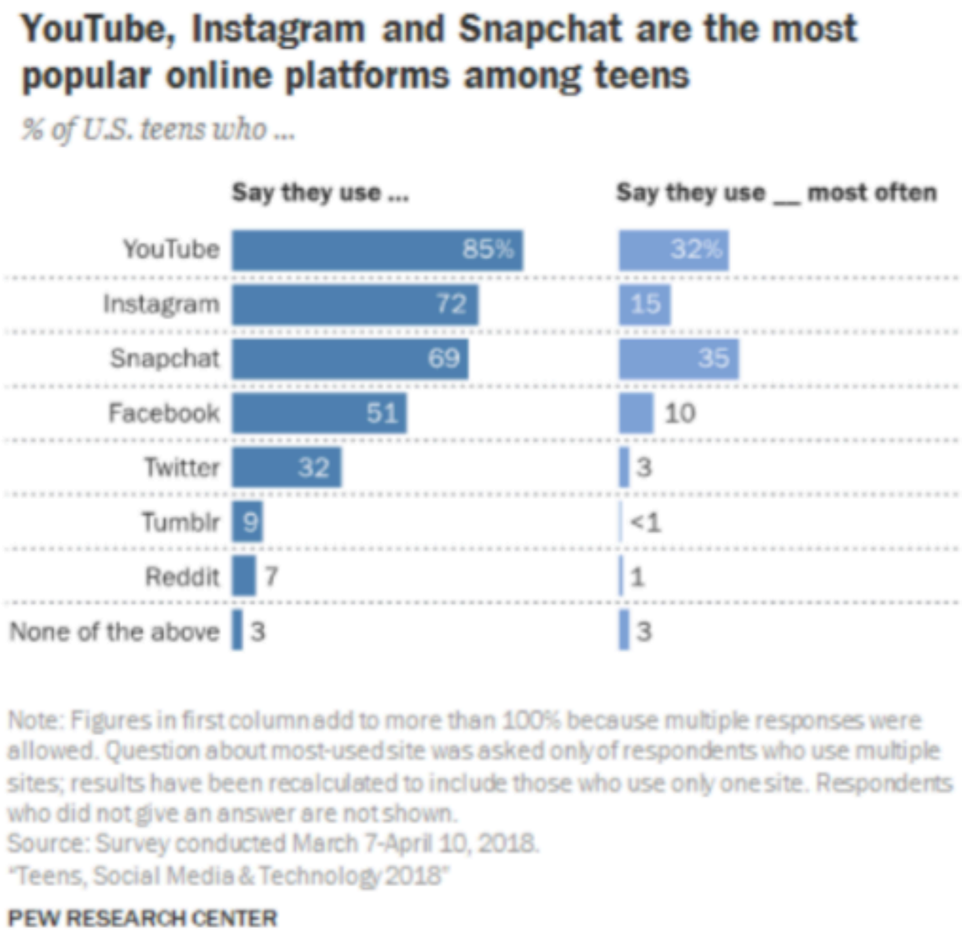
These basic computational principles have been known for a long time, and in fact, gamblers thought about these ideas a lot. Then mathematicians investigated the principles.

Famous problem: will the probability of **at least one six** in **four** throws of a die be equal to prob of **at least a double six** in 24 throws of a pair of dice.

Note: single = die, plural = dice:



Section 1.2: Exact Calculations, or Bound?



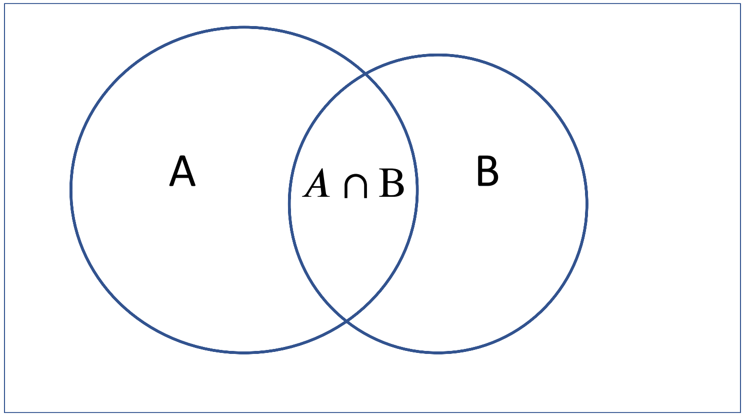
Recall #3 about FB or Twitter. What was the answer? What can you say about the chance that a randomly selected teen used FB or Twitter (not necessarily most often)?

Bounds

When we get some information about the outcome or event whose probability we want to figure out, our outcome space reduces, incorporating that information.

$P(A \cup B)$ for mutually exclusive events

Bounds on probabilities of unions and intersections when events are ***not*** mutually exclusive.



$P(A) = 0.7,\ P(B) = 0.5$

$\_\_\_ \leq P(A \cup B) \leq \_\_\_$

$\_\_\_ \leq P(A \cap B) \leq \_\_\_$

Example with bounds

Let A be the event that you catch the bus to class instead of walking, P(A) = 70%

Let B be the event that it rains, P(B) = 50%

What is the chance of **at least** one of these two events happening?

What is the chance of **both** of them happening?

Exercise: what about if we have 3 events?

Let A be the event that you catch the bus to class instead of walking, P(A) = 70%

Let B be the event that it rains, P(B) = 50%

Let C be the event that you are on time to class, P(C) = 10%

What is the chance of **at least** one of these three events happening?

What is the chance of **all three** of them happening?

Notation: Intersections and Unions

When two events A ***and*** B ***both*** happen, we call this the ***intersection*** of A and B and write it as

$$A \ and \ B = A \cap B \text{ (also written as } AB\text{)}$$

When either A ***or*** B happens, we call this the ***union*** of A and B and write it as

$$A \ or \ B = A \cup B$$

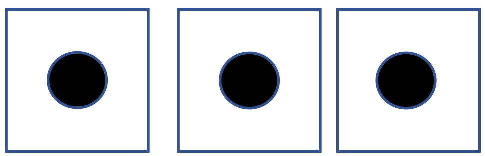
If two events A and B ***cannot both occur*** at the same time, we say that they are *mutually exclusive* or *disjoint*.

$$A \cap B = \varnothing$$

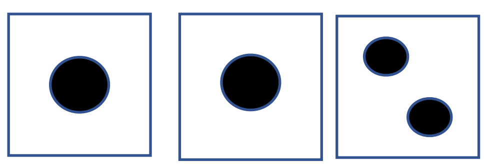
Example of complements

Roll a die 3 times, let A be the event that we roll an ace **each** time.

A^C = **not** A, or not ***all*** aces. It is **not equal** to "never an ace".

A = 

What about "not A"? Here is an example of an outcome in that set.



§1.3: Fundamental Rules

Also called "Axioms of probability", first laid out by Kolmogorov

Recall Ω, the outcome space. Note that Ω can be finite or infinite.

<u>First, some notation:</u>

Events are denoted (usually) by A, B, C …

Recall that Ω is itself an event (called the ***certain*** event) and so is the empty set (denoted $\varnothing$, and called the ***impossible*** event or the *empty set*)

The ***complement*** of an event A is everything ***else*** in the outcome space (all the outcomes that are *not* in A). It is called "not A", or the complement of A, and denoted by A^c

The Axioms of Probability

Think about probability as a ***function*** on ***events***, so put in an event A, and output $P(A)$, a number between 0 and 1 satisfying the axioms below.

Formally: $A \subseteq \Omega,\ P(A) \in [0,1]$ such that

1. For every event $A \subseteq \Omega$, we have $P(A) \geq 0$
2. The outcome space is certain, that is: $P(\Omega) = 1$
3. Addition rule: If two events are mutually exclusive, then the probability of their union is the sum of their probabilities:

$$A \cap B = \varnothing \Rightarrow P(A \cup B) = P(A) + P(B)$$

The third axiom is actually more general and says: If we have infinitely many events that are *mutually exclusive* (no pair overlap), then the chance of their union is the sum of their probabilities.

Consequences of the axioms

1. **Complement rule:** $P\left(A^c\right) = 1 - P(A)$

What is the probability of **not** rolling a pair of sixes in a roll of a pair of dice?

2. **Difference rule:** If $B \subseteq A$, then $P(A \setminus B) = P(A) - P(B)$ where $\mathrm{A} \setminus \mathrm{B}$ refers to the *set difference between* A *and* B*, that* is, all the outcomes that are A but not in B.

3. **Boole's (and Bonferroni's) inequality:** generalization of the fact that the probability of the union of A and B is ***at most*** the sum of the probabilities.

Exercise: De Morgan's Laws

Exercise: Try to show these using Venn diagrams and shading:

1. $(A \cap B)^c = A^c \cup B^c$

2. $(A \cup B)^c = A^c \cap B^c$

Examples

- What is the probability that the top card in a 52 card deck is a queen **and** the bottom card is a queen?

- What is the probability that the top card in a 52 card deck is a queen **or** the bottom card is a queen?