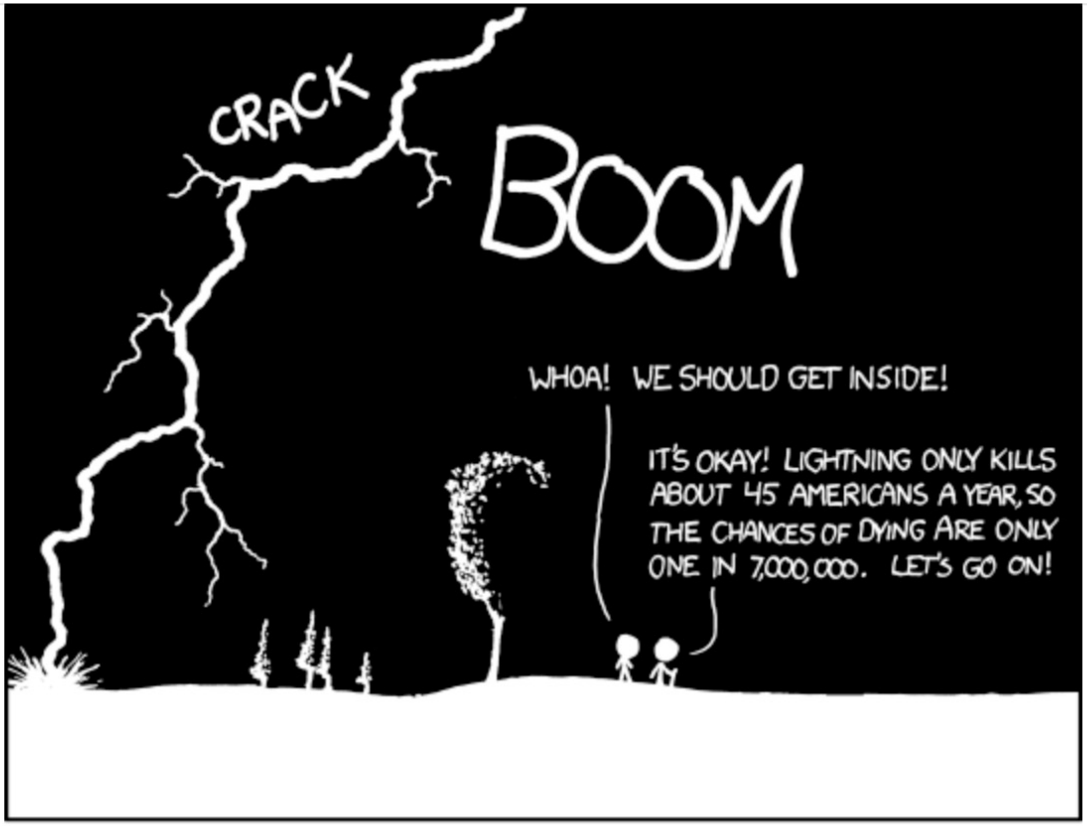
Stat 88: Probability and Statistics in Data Science



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

https://xkcd.com/795/

Lecture 3: 1/25/2022

Axioms of Probability, Intersections,

Sections 1.3, 2.1

Shobhana M. Stoyanov

Agenda

- Section 1.3: Fundamental Rules (the Axioms of Probability)
  - Notation
  - Axioms
  - Consequences of the axioms
  - De Morgan's Law

- Section 2.1: The Probability of Intersections
  - Conditioning
  - Multiplication rule

So far:

- Defined *random experiments*, and their ***outcomes, the outcome space*** (aka ***the sample space*** Ω), ***events, complements of events, the certain event*** (Ω), ***the impossible event*** $\emptyset$
- If all the possible outcomes are ***equally likely***, then each outcome has probability $1/n$, where $n = \#(\Omega)$ and $P(A) = \frac{\#(A)}{\#(\Omega)}, \mathrm{A} \subseteq \Omega$
- Sum of the probabilities of all the distinct outcomes should add to 1
- $0 \leq P(A) \leq 1, \mathrm{A} \subseteq \Omega$
- Venn diagrams
- A *distribution* of the outcomes over different categories is when each outcome appears in one and only one category.
- When we get some information about the outcome or event whose probability we want to figure out, our outcome space ***reduces***, incorporating that information. We now call the probabilities that we compute ***conditional probabilities***

Notation review: Intersections and Unions

- When two events A ***and*** B ***both*** happen, we call this the ***intersection*** of A and B and write it as

$$A \text{ and } B = \mathrm{A} \cap B \text{ (also written as } AB\text{)}$$

- When either A ***or*** B happens, we call this the ***union*** of A and B and write it as

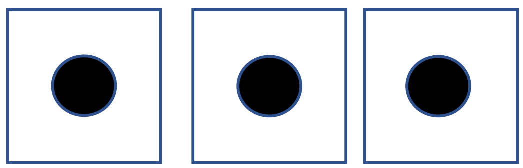
$$A \text{ or } B = A \cup B$$

- If two events A and B ***cannot both occur*** at the same time, we say that they are *mutually exclusive* or *disjoint*.

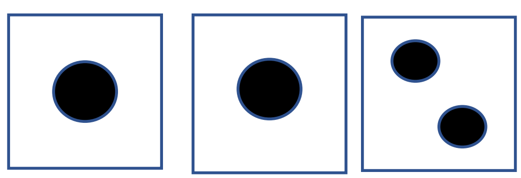
$$\mathrm{A} \cap B = \emptyset$$

Example of complements

- Roll a die 3 times, let A be the event that we roll an ace **each** time.
- A^C = **not** A, or not *all* aces. It is **not equal** to “never an ace”.
- A =

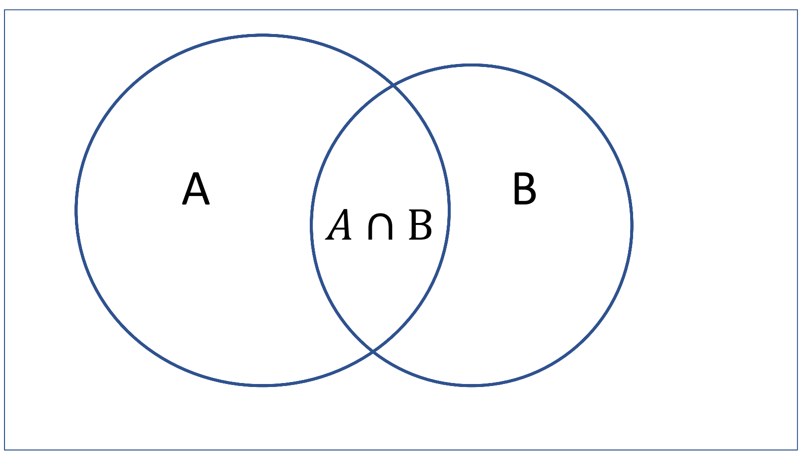


- What about “not A”? Here is an example of an outcome in that set.



Bounds

- When we get some information about the outcome or event whose probability we want to figure out, our outcome space reduces, incorporating that information.
- $P(A \cup B)$ for mutually exclusive events
- **Bounds** on probabilities of unions and intersections when events are ***not*** mutually exclusive.



- P(A) = 0.7, P(B) = 0.5
- $\_\_\_ \leq P(A \cup B) \leq \_\_\_$
- $\_\_\_ \leq P(A \cap B) \leq \_\_\_$

Exercise from Thursday

- A ten-sided fair die is rolled twice:

  - If the first roll lands on 1, what is the chance that the second roll lands on a number bigger than 1?

  - Find the probability that the second number is greater than the ***twice*** the first number.

Example with bounds

- Let A be the event that you catch the bus to class instead of walking, P(A) = 70%
- Let B be the event that it rains, P(B) = 50%
- Let C be the event that you are on time to class, P(C) = 10%
- What is the chance of **at least one** of these three events happening?

- What is the chance of **all three** of them happening?

§1.3: Fundamental Rules

- Also called "Axioms of probability", first laid out by Kolmogorov
- Recall Ω, the outcome space. Note that Ω can be finite or infinite.

- <u>First, some notation:</u>
- Events are denoted (usually) by $\mathrm{A}, \mathrm{B}, \mathrm{C} \ldots$

- Recall that Ω is itself an event (called the ***certain*** event) and so is the empty set (denoted $\emptyset$, and called the ***impossible*** event or the *empty set*)

- The *complement* of an event A is everything ***else*** in the outcome space (all the outcomes that are *not* in A). It is called "not A", or the complement of A, and denoted by A^c

The Axioms of Probability

Think about probability as a ***function*** on ***events***, so put in an event A, and output $P(A)$, a number between 0 and 1 satisfying the axioms below.

Formally: $A \subseteq \Omega, P(A) \in [0,1]$ such that

1. For every event $A \subseteq \Omega$, we have $P(A) \geq 0$
2. The outcome space is certain, that is: $P(\Omega) = 1$
3. Addition rule: If two events are mutually exclusive, then the probability of their union is the sum of their probabilities:

$$A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$$

The third axiom is actually more general and says: If we have infinitely many events that are *mutually exclusive* (no pair overlap), then the chance of their union is the sum of their probabilities.

Consequences of the axioms

1. **Complement rule**: $P(A^c) = 1 - P(A)$

What is the probability of **not** rolling a pair of sixes in a roll of a pair of dice?

2. **Difference rule**: If $B \subseteq A$, then $P(A \setminus B) = P(A) - P(B)$ where $A \setminus B$ refers to the *set difference between* A *and* B*, that* is, all the outcomes that are A but not in B.

3. **Boole's (and Bonferroni's) inequality:** generalization of the fact that the probability of the union of A and B is ***at most*** the sum of the probabilities.

Exercise: De Morgan's Laws

- Exercise: Try to show these using Venn diagrams and shading:

1. $(A \cap B)^c = A^c \cup B^c$

2. $(A \cup B)^c = A^c \cap B^c$

Exercise 1.4.5

- Here's a question from Quora: "If a student applies to ten colleges with a 20% chance of being accepted to each, what are the chances that he will be accepted by at least one college?" Without making any further assumptions, what can you say about this chance?

Probability of an intersection

- Say we have three colored balls in an urn (red, blue, green), and we draw two balls ***without*** replacement.
- Find the probability that the first ball is red, and the second is blue
- Write down the outcome space and compute the probability

- We can also write it down in sequence: P(first red, then blue) = P(first drawing a red ball)P(second ball is blue, given 1st was red)

Multiplication rule

- Conditional probability written as P(B|A), read as "the probability of the event B, given that the event A has occurred"
- Chance that two things will both happen is the chance that the first happens, *multiplied* by the chance that the second will happen *given* that the first has happened.

- Let $A, B \subseteq \Omega, P(A) > 0, P(B) > 0$

- Multiplication rule:

$$P(AB) = P(A|B)\times P(B)$$

$$P(AB) = P(BA) = P(B)\times P(A|B)$$

Multiplication rule

$$P(AB) = P(A|B) \times P(B)$$

- Ex.: Draw a card at random, from a standard deck of 52
  - P(King of hearts) =?
- Draw **2** cards one by one, **without** replacement.
  - P(1st card is K of hearts)=
  - P(2nd card is Q of hearts| 1st is K of hearts) =
  - P(1st card is K of hearts AND 2nd is Q of hearts) =

Addition rule:

- Addition rule: If *A* and *B* are ***mutually exclusive*** events, then the probability that **at least one** of the events will occur is the sum of their probabilities:

$$P(A \cup B) = P(A) + P(B)$$

- If they are not mutually exclusive, does this still hold?
- How do we write the event that "at least one of the events *A* or *B* will occur? How do we draw it?

Inclusion-Exclusion Formula (general addition rule)

- $P(A \cup B) = P(A) + P(B) – P(A \cap B) = P(A) + P(B) – P(AB)$

- $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

  $– P(AB) – P(AC) – P(BC)$

  $+ P(ABC)$

- Of course, if A and B (or A and B and C) *don't* intersect, then the general addition rule becomes the **simple** addition rule of

  $P(A \cup B) = P(A) + P(B)$, or

  $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Examples

- What is the probability that the top card in a 52 card deck is a queen **and** the bottom card is a queen?

- What is the probability that the top card in a 52 card deck is a queen **or** the bottom card is a queen?

Examples

- Roll a pair of dice. What is the chance of rolling at least one 6? Compute this in two ways: using inclusion-exclusion, and using the complement rule

De Méré's paradox:

Find the probability of at least 1 six in 4 throws of a fair die, and at least a double six in 24 throws of a pair of dice.

Exercise for Thursday

- Deal 5 cards from the top of a well shuffled deck. What is the probability that all are hearts?

- Deal 5 cards, what is the chance that they are all the same suit? (flush)

§ 2.2: Symmetry in Simple Random Sampling

- One of the topics we will revisit many times is ***simple random sampling***.
- Sampling **without** replacement, each time with equally likely probabilities
- Example to keep in mind: dealing cards from a deck

- Sampling with replacement: We keep putting the sampled outcomes back before sampling again. (Can do this to simulate die rolls.)

- Need to count number of possible outcomes from repeating an action such as sampling, will use the product rule of counting.

Product rule of counting

- If a set of actions (call them A_1, A_2, …, A_n) can result, respectively, in k_1, k_2, …, k_n possible outcomes, then the entire set of actions can result in:

$$k_1 \times k_2 \times k_3 \times \cdots \times k_n \text{ possible outcomes}$$

- For example: toss a coin twice. Each toss can have 2 possible outcomes, therefore 2 tosses can have 4 possible outcomes.
- So we can count the outcomes for *each* action and multiply these counts to get the number of possible sequences of outcomes.

How many ways to arrange…

- Consider the box that contains O R A N G E:
- How many ways can we rearrange these letters?

- Now say we only want to choose **2 letters** out of the six: __ __

Symmetries in cards

- Deal 2 cards from top of the deck.
    - How many possible sequences of 2 cards?
    - What is the chance that the second card is red?

- $P(5^{th}$ card is red$)$

- $P(R_{21} \cap R_{35})$ = (write it using conditional prob)

- $P(7^{th}$ card is a queen$)$

- $P(B_{52} \mid R_{21}R_{35})$

§ 2.3: Bayes' Rule

- I have two containers: a jar and a box. Each container has five balls: The jar has three red balls and two green balls, and the box has one red and four green balls.
- Say I pick one of the containers at random, and then pick a ball at random. What is the chance that I picked the box, if I ended with a red ball?