Probability and Mathematical Statistics in Data Science

Lecture 11: 4.3: Poisson Distribution

Other Examples of Waiting Times

- The occurrence of events over time. Events of interest might be visits to a particular website, email messages sent to a particular address or accidents in an industrial facility.

- These sort of events are known to follow the **Poisson Distribution.**

Poisson Distribution

- The binomial, hypergeometric, and negative binomial distributions were all derived by starting with an experiment consisting of trials or draws and applying the laws of probability to various outcomes of the experiment. There is no simple experiment on which the Poisson distribution is based on.

A discrete random variable X is said to have a **Poisson distribution** with parameter μ ($\mu > 0$) if the pmf of X is

$$p(x; \mu) = \frac{e^{-\mu} \cdot \mu^x}{x!} \quad x = 0, 1, 2, 3, \ldots$$

Example

- Let X denote the number of creatures of a particular type captured in a trap during a given time period.

- Suppose that X has a Poisson distribution with μ =4.5, so on average traps will contain 4.5 creatures. Then the probability that a trap contains exactly five creatures is

$$P(X=5)=\frac{e^{-4.5}(4.5)^5}{5!}=0.1708$$

Example continued

- The probability that a trap has at most five creatures is

$$P(X \leq 5) = \sum_{x=0}^{5} \frac{e^{-4.5}(4.5)^x}{x!} = e^{-4.5}\left[1 + 4.5 + \frac{(4.5)^2}{2!} + \cdots + \frac{(4.5)^5}{5!}\right] = .7029$$

Question

$$P_k(t) = e^{-\alpha t} \cdot (\alpha t)^k / k!$$

The number of requests for assistance received by a towing service is a Poisson process with rate **α= 4** per hour.

a. Compute the probability that exactly ten requests are received during a particular 2-hour period.

b. If the operators of the towing service take a 30-min break for lunch, what is the probability that they do not miss any calls for assistance?

c. How many calls would you expect during their break?

Relationship between Poisson and Binomial distributions

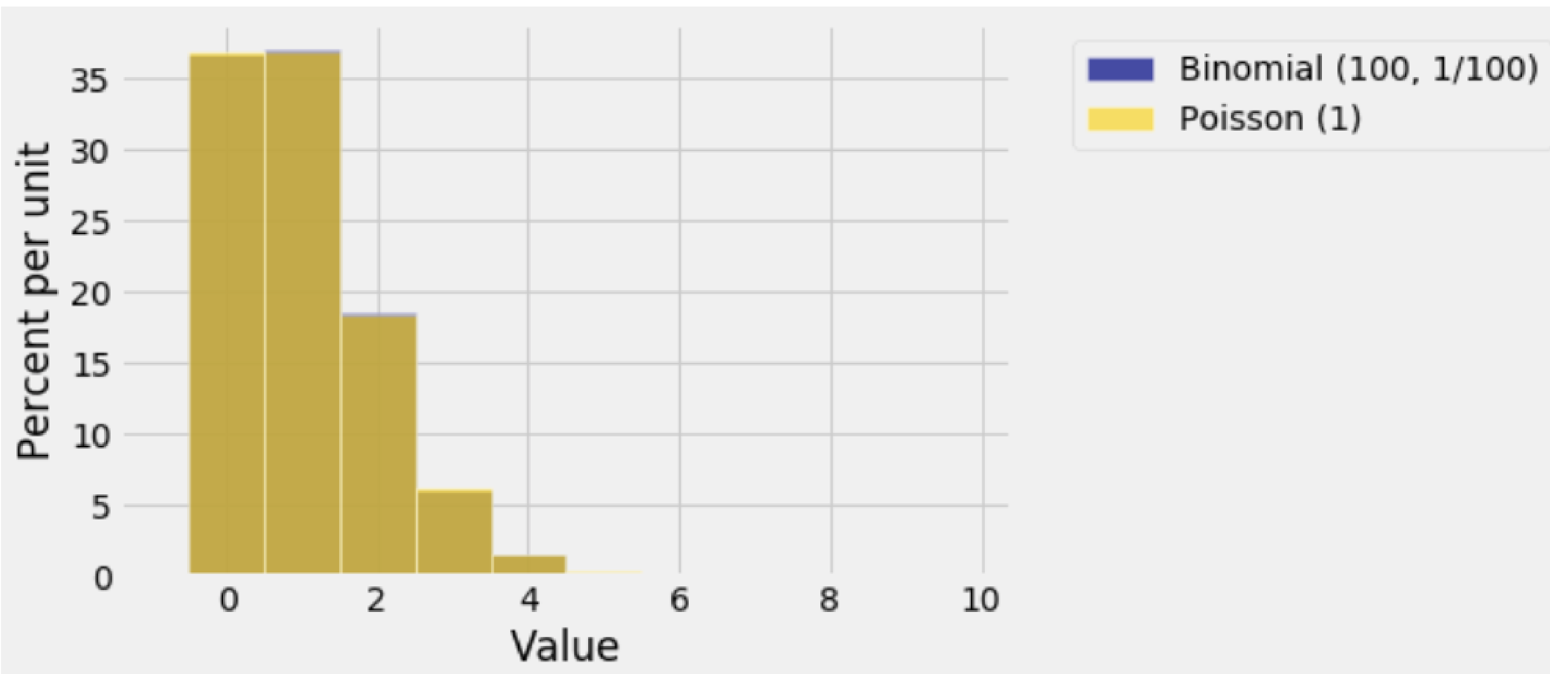
- Suppose that in the binomial pmf $b(x; n; p)$, we let $n \to \infty$ and $p \to 0$ in such a way that np approaches a value $\mu > 0$. Then

- $b(x; n; p) \to p(x; \mu)$.

- So in any binomial experiment in which n is large and p is small, the Binomial can be approximated by Poisson Distribution with parameter $\mu = np$.

Relationship between Poisson and Binomial distributions

- ***The Law of Small Numbers***: when $\boldsymbol{n}$ is large and $\boldsymbol{p}$ is small, the binomial (n,p) distribution is *well approximated* by the Poisson(μ) distribution where $\boldsymbol{\mu = np}$.



Example

A typesetter, on the average makes one error in every 500 words typeset. A typical page contains 300 words. What is the probability that there will be no more than two errors in five pages?

X is the number of errors in five pages

X - Bin(1500; 1/500)

Exact solution

$$P(X \leq 2) = \sum_{x=0}^{2} \binom{1500}{x} \left(\frac{1}{500}\right)^{x} \left(\frac{499}{500}\right)^{1500-x} = .4230$$

With Poisson Approximation $\mu = np = 3$

$$P(X \leq 2) \approx e^{-3} + 3e^{-3} + \frac{3^2 e^{-3}}{2} = .4232$$

Sums of independent Poisson random variables

- If X and Y are random variables such that X and Y are independent

- X has the Poisson(μ) distribution, and

- Y has the Poisson(λ)distribution,

- then the sum $S=X+Y$ has the Poisson ($\mu+\lambda$) distribution.

Exercise 4.5.8

In the first hour that a bank opens, the customers who enter are of three kinds: those who only require teller service, those who only want to use the ATM, and those who only require special services (neither the tellers nor the ATM). Assume that the numbers of customers of the three kinds are independent of each other, and also that:

- the number that only require teller service has the Poisson (6) distribution,
- the number that only want to use the ATM has the Poisson (2) distribution, and
- the number that only require special services has the Poisson (1) distribution.

Suppose you observe the bank in the first hour that it opens. In each part below, find the chance of the event described.

a) 12 customers enter the bank

b) more than 12 customers enter the bank

c) customers do enter but none requires special services