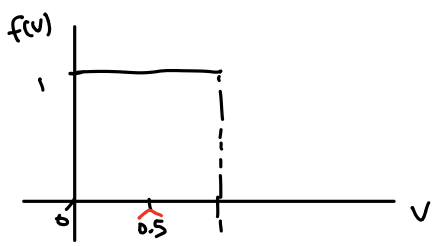
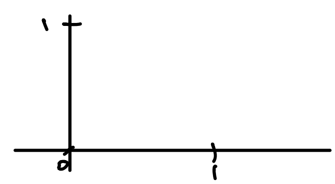
STAT 88: Lecture 31

Contents

Warm up: Let V have density

$$f(v) = \begin{cases} 1 & \text{if } 0 < v < 1 \\ 0 & \text{otherwise} \end{cases}$$





(a) Find the cdf of V.

(b) Find $E(V)$ and $\text{Var}(V)$.

Last time

Density:

For discrete random variables, such as $X \sim \text{Binom}(n, p)$, the probability mass function $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, or the cdf $F(k) = P(X \leq k)$, describes the distribution.

For continuous random variables, such as $Z \sim \mathcal{N}(0, 1)$, the density $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, or the cdf $P(Z \leq z) = \Phi(z)$, describe the distribution.

A function f is called density if it is always nonnegative and integrates to 1.

If X is a continuous random variable, then f is a density of X if

$$P(a < X \leq b) = \int_a^b f(x)dx.$$

Expectation and Variance:

If a continuous random variable X has density f,

$$E(X) = \int_{-\infty}^{\infty} x f(x)dx.$$

For any function g, we have

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

In particular, this shows

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx.$$

The variance and SD are then given by

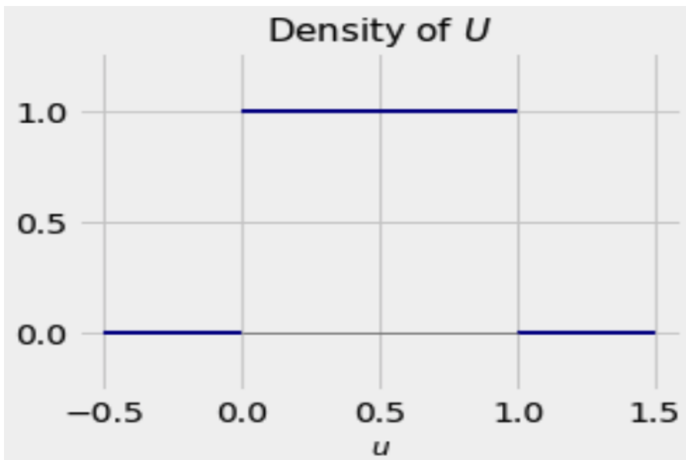
$$\text{Var}(X) = E(X^2) - (EX)^2 \text{ and } \text{SD}(X) = \sqrt{\text{Var}(X)}.$$

10.2. Expectation and Variance

Uniform$(0,1)$ **Distribution**

A random variable U has the uniform distribution on the unit interval $(0,1)$ if

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



For $0 < u_1 < u_2 < 1$, what is $P(u_1 < U < u_2)$?

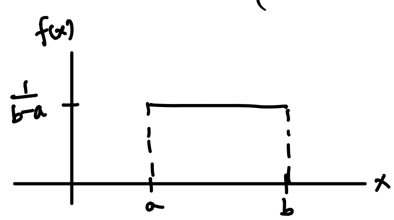
Find and sketch the cdf of f, $F(x)$:

Find $E(U)$ and $\mathrm{Var}(U)$.

Uniform(a, b) **Distribution**

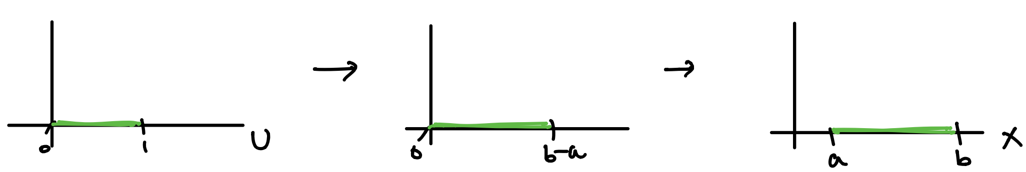
For $a < b$, the random variable X has the uniform distribution on the interval (a, b) if

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$



For $a < x_1 < x_2 < b$, what is $P(x_1 < X < x_2)$?

What function $X = g(U)$ stretches and shifts?



Find $E(X)$ and $\mathrm{Var}(X)$.

Example: (Exercise 10.5.2) A class starts at 3:10 p.m. Seven students in the class arrive at random times $T_1, T_2, \ldots, T_7$ that are i.i.d. with the uniform distribution on the interval 3:07 to 3:12.

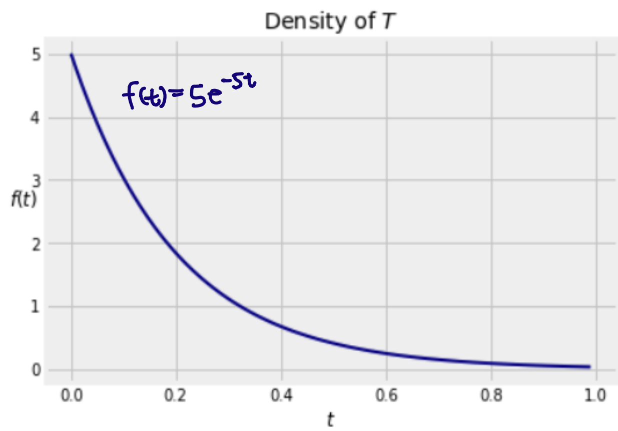
(a) Find $E(T_1)$.

(b) What is the chance that all seven students arrive before 3:10?

(c) Let $X = \max(T_1, T_2, \ldots, T_7)$ be the time when the last of the seven students arrives. Find the cdf of X.

10.3. Expectation and Variance

For $\lambda > 0$, a random variable T has the exponential distribution with rate λ (written $T \sim \text{Exp}(\lambda)$), if the density of T is $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.



The exponential distribution is often used as a model for random lifetimes.

Example: Think of T as the lifetime of an object like a lightbulb and λ as a rate that a lightbulb burns out, e.g. $\lambda = 2$ bulbs/decade. Then the chance a bulb dies in 3 years is

$$
\begin{aligned}
P(T \leq 0.3) &= \int_0^{0.3} 2e^{-2t} dt = -e^{-2t}|_{t=0}^{3} \\
&= -e^{-0.6} + 1 \\
&\approx 0.45.
\end{aligned}
$$

↑ 3 years

CDF and Survival Function The cdf $P(T \leq t)$ indicates the chance that the lightbulb dies before time t:

$$
\begin{aligned}
F(t) = P(T \leq t) &= \int_0^t \lambda e^{-\lambda s} ds \\
&= -e^{-\lambda s}|_{s=0}^{t} \\
&= 1 - e^{-\lambda t}.
\end{aligned}
$$

The survival function $S(t)$ is the chance that the lightbulb survives past time t:

$$S(t) = P(T > t) = 1 - F(t) = e^{-\lambda t}.$$

Memoryless Property Lets find the chance that T survives time $t + s$, given that it survives time t:

$$P(T > t + s \mid T > t) =$$

The exponential distribution is the continuous analog to the geometric distribution. Both have the *memoryless* property.

Mean and SD Let $T \sim \text{Exp}(\lambda)$. Then

$$E(T) = \int_0^\infty t\lambda e^{-\lambda t} dt =$$

The bigger λ is the sooner the bulb dies.

$$E(T^2) = \int_0^\infty t^2 \lambda e^{-\lambda t} dt =$$

$$\text{Var}(T) = E(T^2) - (ET)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}, \quad \text{SD}(T) = \sqrt{\text{Var}(T)} = \frac{1}{\lambda}.$$

Example: Let $X \sim \text{Exp}(\lambda)$ with $E(X) = 10$. Find $P(X > 25 \,|X > 10)$.