STAT 88: Lecture 12

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Sec 5.1 The expectation of a random variable X, denoted $E(X)$, is the average of the possible values of X weighted by their probabilities:

$$E(X) = \sum_{\text{all } x} xP(X = x).$$

(Bernoulli RV)

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

Then

$$E(X) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

(Uniform distribution) X has a uniform distribution on $\{1, 2, \ldots, n\}$ if

$$P(X = k) = \frac{1}{n} \text{ for } k = 1, \ldots, n.$$

Then

$$E(X) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{n+1}{2}.$$

(Poisson distribution) X has a Poisson distribution if

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!} \text{ for } k = 0, 1, \ldots.$$

Then

$$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\mu}\mu^k}{k!} = \mu.$$

<u>Sec 5.2</u> Let $Y = g(X)$ be a function of the random variable X. Then

$$E(Y) = E(g(X)) = \sum_{\text{all } x} g(x)P(X = x).$$

Warm up: (Exercise 5.7.1) Let X have the distribution displayed in the table below.

x	-2	-1	0	1
$P(X = x)$	0.2	0.25	0.35	0.2

(c) Find $E|X - 1|$.

(d) Find $E((X - 1)^2)$.

5.2. Functions of Random Variables (Continued)

Recall that a random variable is a function on the outcome space Ω, $X : \Omega \to \mathbb{R}$.

Example: Flip a fair coin twice. $\Omega = \{HH, HT, TH, TT\}$. Let $X = \#$ heads in 2 coin tosses. Then $X(HH) = 2$.

Now if $Y = g(X) = X^2$, this is also a function of the outcome space

$$Y(HH) = (X(HH))^2 = 4.$$

So a function of a random variable is itself a random variable.

Joint Distribution Suppose two draws are made at random without replacement from a population that has five elements labeled 1,2,**2**,3,3. Define the following random variables:

- X_1 is the number on the first draw
- X_2 is the number on the second draw

The pair (X_1, X_2) is a random variable and we describe its distribution in a table

$$P(X_1 = 1, X_2 = 1) = P(X_1 = 1)P(X_2 = 1|X_1 = 1) = \frac{1}{5} \cdot 0 = 0.$$

$P(X_1 = 1, X_2 = 2)$?

This fills out the table:

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$
$X_1 = 1$	0	$\frac{2}{20}$	$\frac{2}{20}$
$X_1 = 2$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$
$X_1 = 3$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{2}{20}$

Find $P(X_1 + X_2 = 4)$?

Marginal Distribution Note that

$$P(X_1 = 1) = P(X_1 = 1, X_2 = 1) + P(X_1 = 1, X_2 = 2) + P(X_1 = 1, X_2 = 3).$$

So we can recover the distribution of X_1 from the joint distribution.

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	**Dist of X_1**
$X_1 = 1$	0	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20} = \frac{1}{5}$
$X_1 = 2$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{8}{20} = \frac{2}{5}$
$X_1 = 3$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{2}{20}$	$\frac{8}{20} = \frac{2}{5}$
Dist of X_2				

You can get the distribution of X_2 by summing along columns.

Are X_1 and X_2 independent?

X_1 and X_2 are independent if

$$P(X_1 = a, X_2 = b) = P(X_1 = a)P(X_2 = b)$$

for all cells in the joint table.

Expectations of Functions We can find the expectation of any function g of two random variables X and Y by extending our method for finding the expectation of a function of X.

- Take a cell of the joint distribution table of X and Y. This corresponds to one possible value (x, y) of the pair (X, Y).
- Apply the function g to get $g(x, y)$.
- Weight this by the probability in the cell, to get the product $g(x, y)P(X = x, Y = y)$.
- Add these products over all the cells of the table.

Example: Find $E|X_1 - X_2|$.

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$
$X_1 = 1$	0	$\frac{2}{20}$	$\frac{2}{20}$
$X_1 = 2$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$
$X_1 = 3$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{2}{20}$

Example: A joint distribution for two random variables M and S is given below.

	$M = 2$	$M = 3$
$S = 2$	0	$\frac{1}{3}$
$S = 1$	$\frac{1}{3}$	$\frac{1}{3}$

Find $E(M)$.

5.3. Method of Indicators

Preliminary: Additivity of Expectation No matter what the joint distribution of X and Y is, we have

$$E(X+Y) = E(X) + E(Y).$$

This additivity is one of the most important properties of expectation, because it is true whether the random variables are dependent or independent.

Furthermore, for any constants a and b, the linearity also holds:

$$E(aX + bY) = aE(X) + bE(Y).$$

Method of Indicators to find $E(X)$:

Key idea Counting the number of successful trials is the same as adding zeros and ones.

Example: A success is blue, and failure non blue

B R R G B R B B

1 0 0 0 1 0 1 1

$$\#\text{blue} = 1+0+0+0+1+0+1+1 = 4.$$

Recall (Bernoulli (indicator) random variable)

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Binomial X is the number of successes in n independent trials all with the same probability p of success. To find $E(X)$, write X as a sum of n indicators.

$$X = I_1 + I_2 + \cdots I_n,$$

where

$$I_j = \begin{cases} 1 & \text{if } j\text{th trial is success} \\ 0 & \text{otherwise} \end{cases}$$

prob p

Then

$$E(X) = E(I_1 + I_2 + \cdots + I_n) = E(I_1) + E(I_2) + \cdots + E(I_n) = np.$$

Hypergeometric X is the number of good elements in a simple random sample of size n drawn from a population N elements of which G are good. To find $E(X)$, write X as a sum of n indicators.

$$X = I_1 + I_2 + \cdots I_n,$$

where

$$I_j = \begin{cases} 1 & \text{if } j\text{th draw yields a good element} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E(X) = E(I_1 + I_2 + \cdots + I_n) = E(I_1) + E(I_2) + \cdots + E(I_n) = n\frac{G}{N}.$$

Example: (Exercise 5.7.6) A die is rolled 12 times. Find the expectation of

(a) the number of times the face with five spots appears.