STAT 88: Lecture 11

Contents

Last time

Sec 4.4 The Poisson Distribution

$X = \#$ times an event occurs:

$$X \sim \text{Pois}(\mu).$$

Poisson formula:

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!} \text{ for } k = 0, 1, 2, \ldots.$$

Here μ is the average number of successes.

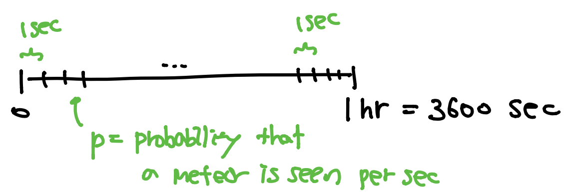
A Poisson distribution is an approximation of Binomial(n, p), when n is large and p is small, with $\mu = np$. The exponential approximation is the key to showing that for large n and small p.

If $X \sim \text{Pois}(\mu)$ and $Y \sim \text{Pois}(\lambda)$ and X, Y are independent, then

$$X + Y \sim \text{Pois}(\mu + \lambda).$$

Example: The number of meteors seen can be modeled as a Poisson distribution because the meteors are independent, the average number of meteors per hour is constant (in the short term). It is expected 5 meteors per hour on average.

Picture



X = # meteors seen per hour ~ pois(5)

Warm up: (Exercise 4.5.11) A courtyard is paved with 100 identical tiles. In an instant of rain, the number of raindrops on each tile has the Poisson (10) distribution independently of all other tiles.

Find the chance that more than 90 tiles have more than 5 raindrops on them.

5.1. Expectation

The expectation of a random variable X, denoted $E(X)$, is the average of the possible values of X weighted by their probabilities:

$$E(X) = \sum_{\text{all } x} xP(X = x).$$

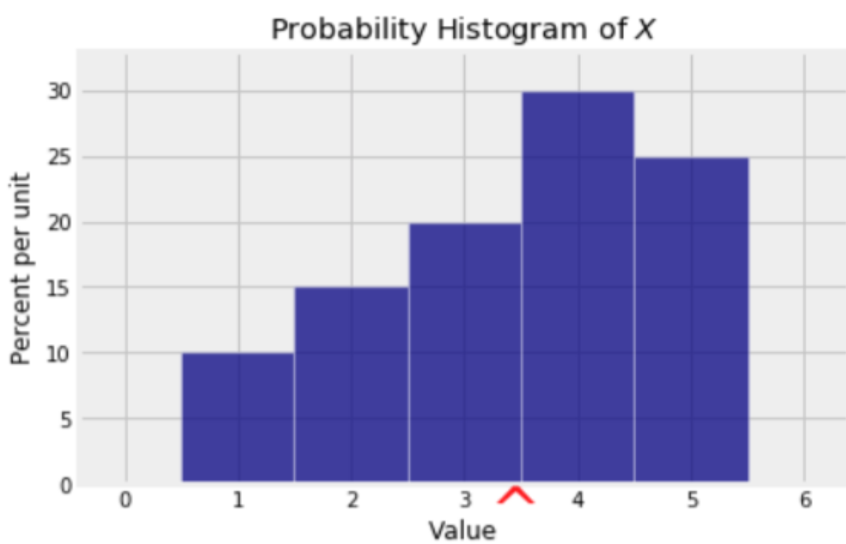
For example, suppose X has the following distribution table.

k	1	2	3	4	5
$P(X = k)$	0.1	0.15	0.2	0.3	0.25

Then

$$E(X) = 1(0.1) + 2(0.15) + 3(0.2) + 4(0.3) + 5(0.25) = 3.45.$$

Here is the probability histogram of X with $E(X)$ marked in red on the horizontal axis.



The object will balance at its center of gravity. The formula for $E(X)$ is the same as the formula for the center of gravity of the object. We say that $E(X)$ is the balance point of the probability histogram of X. This is the sense in which $E(X)$ tells us the location of the distribution of X.

Contstant Suppose a random variable X is actually a constant c, that is, suppose $P(X = c) = 1$. Then the distribution of X puts all its mass on the single value c and $E(X) = c \cdot 1 = c$. We just write $E(c) = c$.

Bernoulli and Indicators A random variable X has the Bernoulli(p) distribution if $P(X = 1) = p$ and $P(X = 0) = 1 - p$. So

$$E(X) = 0 \cdot (1 - p) + 1 \cdot p = p.$$

These random variables are also often called indicators. Let A be an event. Then the indicator of A is the random variable I_A hat has the value 1 if A occurs and 0 if A doesn't occur. i.e.

$$I_A = \begin{cases} 1, & \text{if } A \text{ occurs.} \\ 0, & \text{if } A \text{ doesn't occur.} \end{cases}$$

I_A has the Bernoulli($P(A)$) distribution and

$$E(I_A) = P(A).$$

This shows that every probability is an expectation.

Uniform on $\{1, 2, 3, \ldots, n\}$ Let n be a fixed positive integer. A random variable X has the uniform distribution on the integers 1 through n if X is equally likely to have any of the values 1 through n.

$$E(X) = \sum_i i \cdot \frac{1}{n} = \frac{n+1}{2}.$$

Poisson(μ) Let X have the Pois(μ) distribution. Then

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} k e^{-\mu} \frac{\mu^k}{k!} \\ &= e^{-\mu} \mu \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!} \\ &= e^{-\mu} \mu \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \\ &= e^{-\mu} \mu e^{\mu} \\ &= \mu. \end{aligned}$$

We have seen a RV, X, belong to different distributions depending on the application.

- Uniform
- Binomial
- Hypergeometric
- Geometric
- Poisson

$E(X)$ is the balance point of your data and an important summary statistic.

Properties of Expectation

1. $E(c) = c$ for constant c
2. Linearity: $E(aX + b) = aE(X) + b$

Example: (Exercise 5.7.1) Let X have the distribution displayed in the table below.

x	-2	-1	0	1
$P(X = x)$	0.2	0.25	0.35	0.2

(a) Find $E(X)$.

(b) Find $E(X - 1)$.

Example: (Exercise 5.7.3) A box contains four blank index cards and one that has a gold star on it. Cards are drawn one by one at random without replacement until the gold star appears. Let D be the number of cards drawn.

(a) Find the distribution of D.

(b) Find $E(D)$.

(c) Suppose all five cards were dealt one by one at random without replacement. If you saw the sequence of cards, would you be able to tell whether you were looking at the sequence forwards (that is, in the order in which the cards were drawn) or backwards? If the answer is "no", can you use it to give another justification for the answer to Part b?

5.2. Functions of Random Variables

When we work with random variables, we often want to consider functions of them, e.g.

$$\text{Let } X \sim \text{Unif}\{-1, 0, 1\} \text{ and } Y = X^2.$$

Here is a distribution table for X, with values of Y as well.

$y = x^2$	1	0	1
x	−1	0	1
$P(X = x)$	1/3	1/3	1/3

Then

$$E(X^2) = (-1)^2(\frac{1}{3}) + 0^2(\frac{1}{3}) + 1^2(\frac{1}{3}) = \frac{2}{3}.$$

More generally, let $Y = g(X)$ be a function of the random variable X. Then

$$E(Y) = E(g(X)) = \sum_{\text{all } x} g(x)P(X = x).$$

Example: Let $W = \min\{X, .5\}$. Fill out the table and find $E(W)$.

w = min(x, .5)			
x	−1	0	1
$P(X = x)$	1/3	1/3	1/3