# STAT 88: Lecture 6

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**Last time**

**<u>Sec 2.5</u>** $A$ and $B$ are independent iff

$$P(B|A) = P(B).$$

Then $P(A \cap B) = P(A)P(B)$.

If $A$ and $B$ are independent, non-empty sets, then they must overlap, i.e.

$$P(A \cap B) = P(A)P(B) > 0.$$

In other words, $A$ and $B$ are not mutually exclusive.

Warm up:

(a) You flip a coin 8 times. What is the chance that you get all heads?

(b) Everyone in a class of 100 people flip a coin 8 times. What is the chance that at least one person gets all heads?

# 3.1. Success and Failure

Read Section 3.1 of textbook, Paul the Octopus.

# 3.2. Random Variables

Random Variables (RV) help reduce the amount of writing involved in phrases like "the chance that there are no more than 1 head in three tosses of a coin".

You can instead write:

> Let $X$ be the number of heads in three coin tosses. Find $P(X \leq 1)$.

Formally a random variable $X$ is a function from the outcome space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

| outcome | $X$(outcome) | Probability |
|---|---|---|
| HHH | 3 | 1/8 |
| HHT | 2 | 1/8 |
| HTH | 2 | 1/8 |
| THH | 2 | 1/8 |
| HTT | 1 | 1/8 |
| THT | 1 | 1/8 |
| TTH | 1 | 1/8 |
| TTT | 0 | 1/8 |

$$P(X \leq 1) = P(X=0) + P(X=1) = \boxed{\tfrac{1}{2}}$$

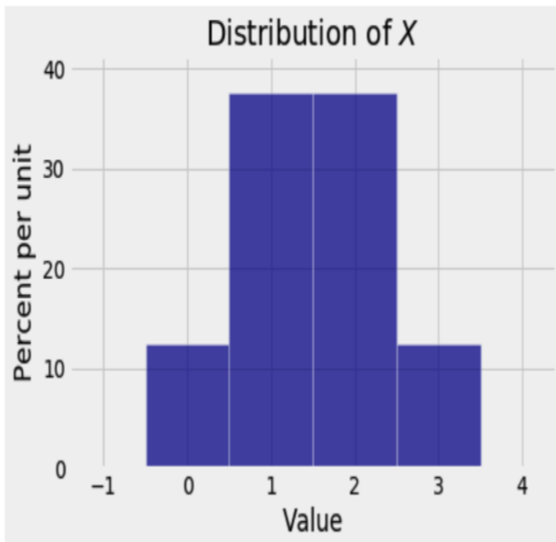
Probability distribution table for $X$, known for short as a distribution table.

| Possible value $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | 1/8 | 3/8 | 3/8 | 1/8 |

**Distribution (pmf)** The probability distribution of a random variable, or distribution for short, is the set of all possible values of the random variables along with all of the corresponding probabilities.

The probabilities in a distribution must add up to 1. The distribution of a random variable is sometimes called a probability mass function, abbreviated to pmf.

**Probability Histogram** The distribution or probability mass function (Pmf) allows us to visualize the probability for each value of $X$.




**Equality** Two RVs can have the same distribution but not be equal. Let $X_1$ be the number of heads and $X_2$ be the number of tails in three tosses. If the outcome of three tosses is HTH, then $X_1(HTH) = 2$ and $X_2(HTH) = 1$ so as functions on the outcome space, $X_1 \neq X_2$. But both RVs have the same distribution.

## 3.3. The Binomial Distribution

A binomial distribution Binomial$(n, p)$ has $n$ independent trials, each with probability $p$ for success.

Example: $X = \#$ heads out of 5 coin tosses of a $p = 1/4$ coin (chance of landing head is $1/4$). Let's find $P(X = 2)$.

Here $n = 5$ independent coin tosses, $p = 1/4$ is chance for heads, and $k = 2$.

First what is the chance that you get HHTTT? HTHTT?

How many permutations of 5 letters abcde? In case of HHTTT we must divide by 2!3!, giving

$$\binom{5}{2} = \frac{5!}{2!3!}.$$

This shows that

$$P(X = 2) = \binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3.$$

What values does $X$ take?

More generally, $X \sim$ Binomial$(n,p)$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Example: (Exercise 3.6.3) Yi likes to bet on "red" at roulette. Each time she bets, her chance of winning is 18/38 independently of all other times. Suppose she bets repeatedly on red. Find the chance that:

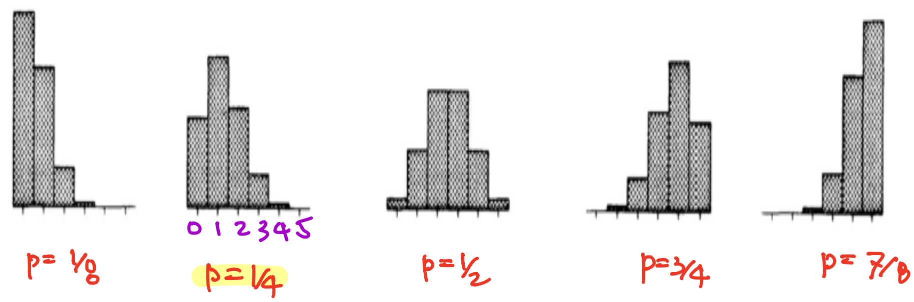
(a) she wins four of the first 10 bets

(b) she wins at most four of the first 10 bets

(c) the third time she wins is on the 10th bet

(d) she needs more than 10 bets to win five times

**Binomial Probabilities in Python** SciPy is a compendium of Python software that is enormously useful in data science. In particular, its stats module contains numerous functions and methods used by data scientists.




In Python:

```
from scipy import stats
import numpy as np
```

```
stats.binom.pmf(2, 5, 1/4)
```

```
0.26367187499999994
```

$\binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3$

```
stats.binom.pmf(np.arange(6), 5, 1/4)
```

```
array([0.23730469, 0.39550781, 0.26367187, 0.08789062, 0.01464844,
       0.00097656])
```

$\binom{5}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{5-x}$ for $x = 0, 1, 2, \cdots, 5$

Why does $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$ ? (so Binomial $(n,p)$ is a distribution)

"Binomial theorem"

$$(p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$\parallel$

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Example: Ten cards are dealt off the top of a well shuffled deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds because:

1. The probability of a trial being successful changes
2. The trials aren't independent
3. There isn't a fixed number of trials
4. More than one of the above